Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

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Mathematics: analysis and approaches

Higher level

Paper 2

35 pages

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Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1, A2***.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

	Correct answer seen	Further working seen	Any FT issues?	Action
1.	$8\sqrt{2}$	5.65685... *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	$0.468111\ldots$ (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.

- If the error leads to an inappropriate value (*e.g*. probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).

- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.

- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a "show that" question correctly, and then in subsequent parts use their incorrect answer rather than the given value.

- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was "Hence".

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and $1,9$ or 1000 and $1,000$ and 1.000.

- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.

- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.

- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x}\times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x}\times e^{3x}-e^{4x}\times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and x^2+x are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

A GDC is required for this paper, but If you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is “first”.

Section A

1. (a) **EITHER**

uses the cosine rule ***(M1)***

$\text{AB}^2 = 5^2 + 5^2 - 2\times5\times5\times\cos1.9$ ***(A1)***

OR

uses right-angled trigonometry ***(M1)***

$\dfrac{\frac{\text{AB}}{2}}{5} = \sin 0.95$ ***(A1)***

OR

uses the sine rule ***(M1)***

$\alpha = \dfrac{1}{2}(\pi - 1.9)(= 0.6207...)$

$\dfrac{\text{AB}}{\sin1.9} = \dfrac{5}{\sin 0.6207...}$ ***(A1)***

THEN

$\text{AB} = 8.1341...$

$\text{AB} = 8.13$ (m) ***A1***

[3 marks]

continued…

Question 1 continued

(b) let the shaded area be A

METHOD 1

Attempt at finding reflex angle **(M1)**

$\hat{AOB} = 2\pi - 1.9 \ (= 4.3831...)$

substitution into area formula **(A1)**

$A = \frac{1}{2} \times 5^2 \times 4.3831... \text{ OR } \left(\frac{2\pi - 1.9}{2\pi}\right) \times \pi\left(5^2\right)$

$= 54.7898...$

$= 54.8$ (m²) ***A1***

METHOD 2

let the area of the circle be A_C and the area of the unshaded sector be A_U

$A = A_C - A_U$ **(M1)**

$A = \pi \times 5^2 - \frac{1}{2} \times 5^2 \times 1.9 \ (= 78.5398... - 23.75)$ **(A1)**

$= 54.7898...$

$= 54.8$ (m²) ***A1***

[3 marks]

Total [6 marks]

2. **METHOD 1**

recognises that $g(x) = \int (3x^2 + 5e^x)\, dx$ ***(M1)***

$g(x) = x^3 + 5e^x (+C)$ ***(A1)(A1)***

Note: Award ***A1*** for each integrated term.

substitutes $x = 0$ and $y = 4$ into their integrated function (must involve $+C$) ***(M1)***

$4 = 0 + 5 + C \Rightarrow C = -1$

$g(x) = x^3 + 5e^x - 1$ ***A1***

METHOD 2

attempts to write both sides in the form of a definite integral ***(M1)***

$\int_0^x g'(t)\, dt = \int_0^x (3t^2 + 5e^t)\, dt$ ***(A1)***

$g(x) - 4 = x^3 + 5e^x - 5e^0$ ***(A1)(A1)***

Note: Award ***A1*** for $g(x) - 4$ and ***A1*** for $x^3 + 5e^x - 5e^0$.

$g(x) = x^3 + 5e^x - 1$ ***A1***

[5 marks]

3. $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) = 0.68$

substitution of $\mathrm{P}(A) \cdot \mathrm{P}(B)$ for $\mathrm{P}(A \cap B)$ in $\mathrm{P}(A \cup B)$ ***(M1)***

$\mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A)\mathrm{P}(B) \ (= 0.68)$

substitution of $3\mathrm{P}(B)$ for $\mathrm{P}(A)$ ***(M1)***

$3\mathrm{P}(B) + \mathrm{P}(B) - 3\mathrm{P}(B)\mathrm{P}(B) = 0.68$ (or equivalent) ***(A1)***

Note: The first two marks are independent of each other.

attempts to solve their quadratic equation ***(M1)***

$\mathrm{P}(B) = 0.2, 1.133\ldots \left(\frac{1}{5}, \frac{17}{15}\right)$

$\mathrm{P}(B) = 0.2 \left(= \frac{1}{5}\right)$ ***A2***

Note: Award ***A1*** if both answers are given as final answers for $\mathrm{P}(B)$.

[6 marks]

4. (a) 0.28 (s) ***A1***

[1 mark]

(b) $\text{IQR} = 0.35 - 0.27 \ (= 0.08)$ (s) ***(A1)***

substituting **their** IQR into correct expression for upper fence ***(A1)***

$0.35 + 1.5 \times 0.08 \ (= 0.47)$ (s)

$0.46 < 0.47$ ***R1***

so 0.46 (s) is not an outlier ***AG***

[3 marks]

(c) **EITHER**

the median is closer to the lower quartile (positively skewed) ***R1***

OR

the distribution is positively skewed ***R1***

OR

the range of reaction times below the median is smaller than the range of reaction times above the median ***R1***

Note: These are sample answers from a range of acceptable correct answers. Award ***R1*** for any correct statement that explains this.

Do not award ***R1*** if there is also an incorrect statement, even if another statement in the answer is correct. Accept a correctly and clearly labelled diagram.

[1 mark]

continued…

Question 4 continued

(d) **EITHER**

the distribution for ‘not sleeping well’ is centred at a higher reaction time ***R1***

OR

the median reaction time after not sleeping well is equal to the upper quartile reaction time after sleeping well ***R1***

OR

75% of reaction times are <0.35 seconds after sleeping well, compared with 50% after not sleeping well ***R1***

OR

the sample size of 9 is too small to draw any conclusions ***R1***

Note: These are sample answers from a range of acceptable correct answers. Accept any relevant correct statement t**hat relates to the median and/or quartiles shown in the box plots**. Do not accept a comparison of means.

Do not award ***R1*** if there is also an incorrect statement, even if another statement in the answer is correct

Award ***R0*** to “correlation does not imply causation”.

[1 mark]

Total [6 marks]

5. (a) recognises the need to find the value of t when $v = 0$ ***(M1)***

$t = 1.5707...\ \left(= \frac{\pi}{2}\right)$

$t = 1.57\ \left(= \frac{\pi}{2}\right)$ (s) ***A1***

[2 marks]

(b) recognises that $a(t) = v'(t)$ ***(M1)***

$t_1 = 2.2627...,\ t_2 = 2.9573...$

$t_1 = 2.26,\ t_2 = 2.96$ (s) ***A1A1***

Note:	Award ***M1A1A0*** if the two correct answers are given with additional values outside $0 \le t \le 3$.

[3 marks]

(c) speed is greatest at $t = 3$ ***(A1)***

$a = -1.8377...$

$a = -1.84$ (m s^{-2}) ***A1***

[2 marks]

Total [7 marks]

6. attempts to express x^2 in terms of y ***(M1)***

$V = \pi\int_h^4 36\left(1 - \frac{(y-4)^2}{16}\right)\mathrm{d}y$ ***A1***

Note: Correct limits are required.

Attempts to solve $\pi\int_h^4 36\left(1 - \frac{(y-4)^2}{16}\right)\mathrm{d}y = 285$ for h ***(M1)***

Note: Award ***M1*** for attempting to solve $36\pi\left(\frac{h^3}{48} - \frac{h^2}{4} + \frac{8}{3}\right) = 285$ or equivalent for h.

$h = 0.7926\ldots$

$h = 0.793$ (cm) ***A2***

[5 marks]

7. (a) (as $\lim_{x \to 0} x^2 = 0$, the indeterminate form $\frac{0}{0}$ is required for the limit to exist)

$\Rightarrow \lim_{x \to 0}\left(\arctan(\cos x) - k\right) = 0$ ***M1***

$\arctan 1 - k = 0 \ (k = \arctan 1)$ ***A1***

so $k = \frac{\pi}{4}$ ***AG***

Note: Award ***M1A0*** for using $k = \frac{\pi}{4}$ to show the limit is $\frac{0}{0}$.

[2 marks]

continued…

Question 7 continued

(b) $\lim_{x \to 0} \frac{\arctan(\cos x) - \frac{\pi}{4}}{x^2} \left(= \frac{0}{0}\right)$

$= \lim_{x \to 0} \frac{\frac{-\sin x}{1 + \cos^2 x}}{2x}$ ***A1A1***

Note: Award ***A1*** for a correct numerator and ***A1*** for a correct denominator.

recognises to apply l'Hôpital's rule again ***(M1)***

$= \lim_{x \to 0} \frac{\frac{-\sin x}{1 + \cos^2 x}}{2x} \left(= \frac{0}{0}\right)$

Note: Award ***M0*** if their limit is not the indeterminate form $\frac{0}{0}$.

EITHER

$= \lim_{x \to 0} \frac{\frac{-\cos x(1 + \cos^2 x) - 2\sin^2 x \cos x}{(1 + \cos^2 x)^2}}{2}$ ***A1A1***

Note: Award ***A1*** for a correct first term in the numerator and ***A1*** for a correct second term in the numerator.

OR

$\lim_{x \to 0} \frac{-\cos x}{2(1 + \cos^2 x) - 4x \sin x \cos x}$ ***A1A1***

Note: Award ***A1*** for a correct numerator and ***A1*** for a correct denominator.

THEN

substitutes $x = 0$ into the correct expression to evaluate the limit ***A1***

Note: The final ***A1*** is dependent on all previous marks.

$= -\frac{1}{4}$ ***AG***

[6 marks]

Total [8 marks]

8. Rachel: $R \sim \mathrm{N}\left(56.5, 3^2\right)$

$\mathrm{P}(R \geq 60) = 0.1216...$ ***(A1)***

Sophia: $S \sim \mathrm{N}\left(57.5, 1.8^2\right)$

$\mathrm{P}(S \geq 60) = 0.0824...$ ***(A1)***

recognises binomial distribution with $n = 5$ ***(M1)***

let N_R represent the number of Rachel's throws that are longer than 60 metres

$N_R \sim \mathrm{B}(5, 0.1216...)$

either $\mathrm{P}(N_R \geq 1) = 0.4772...$ or $\mathrm{P}(N_R = 0) = 0.5227...$ ***(A1)***

let N_S represent the number of Sophia's throws that are longer than 60 metres

$N_S \sim \mathrm{B}(5, 0.0824...)$

either $\mathrm{P}(N_S \geq 1) = 0.3495...$ or $\mathrm{P}(N_S = 0) = 0.6504...$ ***(A1)***

EITHER

uses $\mathrm{P}(N_R \geq 1)\mathrm{P}(N_S = 0) + \mathrm{P}(N_S \geq 1)\mathrm{P}(N_R = 0)$ ***(M1)***

$\mathrm{P}(\text{one of Rachel or Sophia qualify}) = (0.4772... \times 0.6504...) + (0.3495... \times 0.5227...)$

OR

uses $\mathrm{P}(N_R \geq 1) + \mathrm{P}(N_S \geq 1) - 2 \times \mathrm{P}(N_R \geq 1) \times \mathrm{P}(N_S \geq 1)$ ***(M1)***

$\mathrm{P}(\text{one of Rachel or Sophia qualify}) = 0.4772... + 0.3495... - 2 \times 0.4772... \times 0.3495...$

THEN

$= 0.4931...$

$= 0.493$ ***A1***

[7 marks]

Note: ***M*** marks are not dependent on the previous ***A*** marks.

9. (a) $9\times9\times8\times7\times6\times5 \ \left(=9\times{}^{9}P_{5}\right)$ ***(M1)***

$=136080 \ \left(=9\times\dfrac{9!}{4!}\right)$ ***A1***

Note: Award ***M1A0*** for $10\times9\times8\times7\times6\times5 \ \left(={}^{10}P_{6}=151200 \ =\dfrac{10!}{4!}\right)$.

Note: Award ***M1A0*** for ${}^{9}P_{6}=60480$

[2 marks]

(b) **METHOD 1**

EITHER

every unordered subset of 6 digits from the set of 9 non-zero digits can be arranged in exactly one way into a 6-digit number with the digits in increasing order. ***A1***

OR

${}^{9}C_{6}\,(\times1)$ ***A1***

THEN

$=84$ ***A1***

METHOD 2

EITHER

removes 3 digits from the set of 9 non-zero digits and these 6 remaining digits can be arranged in exactly one way into a 6-digit number with the digits in increasing order. ***A1***

OR

${}^{9}C_{3}\,(\times1)$ ***A1***

THEN

$=84$ ***A1***

[2 marks]

Total [4 marks]

Section B

10. (a) (i) 32 (cm) ***A1***

(ii) $h_A(0) = \sin(6) + 27$ ***(M1)***

$= 26.7205...$

$= 26.7$ (cm) ***A1***

[3 marks]

(b) attempts to solve $h_A(t) = h_B(t)$ for t ***(M1)***

$t = 4.0074..., 4.7034..., 5.88332...$

$t = 4.01, 4.70, 5.88$ (weeks) ***A2***

[3 marks]

(c) $h_A(t) - h_B(t) = \sin(2t + 6) + t - 5$ ***A1***

EITHER

for $t > 6$, $t - 5 > 1$ ***A1***

and as $\sin(2t + 6) \geq -1 \Rightarrow h_A(t) - h_B(t) > 0$ ***R1***

OR

the minimum value of $\sin(2t + 6) = -1$ ***R1***

so for $t > 6$, $h_A(t) - h_B(t) = t - 6 > 0$ ***A1***

THEN

hence for $t > 6$, Plant A was always taller than Plant B ***AG***

[3 marks]

continued…

Question 10 continued

(d) recognises that $h_A'(t)$ and $h_B'(t)$ are required **(M1)**

attempts to solve $h_A'(t) = h_B'(t)$ for t **(M1)**

$t = 1.18879...$ and $2.23598...$ OR $4.33038...$ and $5.37758...$ OR $7.47197...$ and $8.51917...$ **(A1)**

Note: Award full marks for $t = \frac{4\pi}{3} - 3, \frac{5\pi}{3} - 3, \left(\frac{7\pi}{3} - 3, \frac{8\pi}{3} - 3, \frac{10\pi}{3} - 3, \frac{11\pi}{3} - 3\right)$.
Award subsequent marks for correct use of these exact values.

$1.18879... < t < 2.23598...$ OR $4.33038... < t < 5.37758...$ OR
$7.47197... < t < 8.51917...$ **(A1)**

attempts to calculate the total amount of time **(M1)**

$3(2.2359... - 1.1887...)$ $\left(= 3\left(\left(\frac{5\pi}{3} - 3\right) - \left(\frac{4\pi}{3} - 3\right)\right)\right)$

$= 3.14$ $(= \pi)$ (weeks) ***A1***

[6 marks]

Total [15 marks]

11. (a) let ϕ be the required angle (bearing)

EITHER

$\phi = 90° - \arctan\frac{1}{2}\ (= \arctan 2)$ ***(M1)***

Note: Award ***M1*** for a labelled sketch.

OR

$\cos\phi = \frac{\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}4\\2\end{pmatrix}}{\sqrt{1}\times\sqrt{20}}\ \left(= 0.4472...,= \frac{1}{\sqrt{5}}\right)$ ***(M1)***

$\phi = \arccos(0.4472...)$

THEN

$063°$ ***A1***

Note: Do not accept $063.4°$ or $63.4°$ or 1.10^c.

[2 marks]

(b) **Method 1**

let $|\boldsymbol{b}_A|$ be the speed of A and let $|\boldsymbol{b}_B|$ be the speed of B

attempts to find the speed of one of A or B ***(M1)***

$|\boldsymbol{b}_A| = \sqrt{(-6)^2 + 2^2 + 4^2}$ or $|\boldsymbol{b}_B| = \sqrt{4^2 + 2^2 + (-2)^2}$

Note: Award ***M0*** for $|\boldsymbol{b}_A| = \sqrt{19^2 + (-1)^2 + 1^2}$ and $|\boldsymbol{b}_B| = \sqrt{1^2 + 0^2 + 12^2}$.

$|\boldsymbol{b}_A| = 7.48...\ (= \sqrt{56})$ (km min^{-1}) and $|\boldsymbol{b}_B| = 4.89...\ (= \sqrt{24})$ (km min^{-1}) ***A1***

$|\boldsymbol{b}_A| > |\boldsymbol{b}_B|$ so A travels at a greater speed than B ***AG***

[2 marks]

continued…

Question 11 continued

Method 2

attempts to use $\text{speed} = \frac{\text{distance}}{\text{time}}$

$\text{speed}_A = \frac{|r_A(t_2) - r_A(t_1)|}{t_2 - t_1}$ and $\text{speed}_B = \frac{|r_B(t_2) - r_B(t_1)|}{t_2 - t_1}$ ***(M1)***

for example:

$\text{speed}_A = \frac{|r_A(1) - r_A(0)|}{1}$ and $\text{speed}_B = \frac{|r_B(1) - r_B(0)|}{1}$

$\text{speed}_A = \frac{\sqrt{(-6)^2 + 2^2 + 4^2}}{1}$ and $\text{speed}_B = \frac{\sqrt{4^2 + 2^2 + 2^2}}{1}$

$\text{speed}_A = 7.48\ldots\left(2\sqrt{14}\right)$ and $\text{speed}_B = 4.89\ldots\left(\sqrt{24}\right)$ ***A1***

$\text{speed}_A > \text{speed}_B$ so A travels at a greater speed than B ***AG***

[2 marks]

continued…

Question 11 continued

(c) attempts to use the angle between two direction vectors formula **(M1)**

$$\cos\theta = \frac{(-6)(4)+(2)(2)+(4)(-2)}{\sqrt{(-6)^2+2^2+4^2}\sqrt{4^2+2^2+(-2)^2}}$$ **(A1)**

$\cos\theta = -0.7637... \left(= -\frac{7}{\sqrt{84}}\right)$ or $\theta = \arccos(-0.7637...)$ $(= 2.4399...)$

attempts to find the acute angle $180° - \theta$ using their value of θ **(M1)**

$= 40.2°$ **A1**

[4 marks]

continued…

Question 11 continued

(d) (i) for example, sets $\boldsymbol{r}_{\boldsymbol{A}}(t_1)=\boldsymbol{r}_{\boldsymbol{B}}(t_2)$ and forms at least two equations ***(M1)***

$19-6t_1=1+4t_2$

$-1+2t_1=2t_2$

$1+4t_1=12-2t_2$

Note: Award ***M0*** for equations involving t only.

EITHER

attempts to solve the system of equations for one of t_1 or t_2 ***(M1)***

$t_1=2$ or $t_2=\frac{3}{2}$ ***A1***

OR

attempts to solve the system of equations for t_1 and t_2 ***(M1)***

$t_1=2$ and $t_2=\frac{3}{2}$ ***A1***

THEN

substitutes their t_1 or t_2 value into the corresponding $\boldsymbol{r}_{\boldsymbol{A}}$ or $\boldsymbol{r}_{\boldsymbol{B}}$ ***(M1)***

$\mathrm{P}(7,3,9)$ ***A1***

Note: Accept $\overrightarrow{\mathrm{OP}}=\begin{pmatrix}7\\3\\9\end{pmatrix}$. Accept 7 km east of O, 3 km north of O and 9 km above sea level.

continued…

Question 11 continued

(ii) attempts to find the value of $t_1 - t_2$ **(*M1*)**

$t_1 - t_2 = 2 - \frac{3}{2}$

0.5 minutes (30 seconds) ***A1***

[7 marks]

continued…

Question 11 continued

(e) **EITHER**

attempts to find $\boldsymbol{r}_B - \boldsymbol{r}_A$ ***(M1)***

$$\boldsymbol{r}_B - \boldsymbol{r}_A = \begin{pmatrix} -18 \\ 1 \\ 11 \end{pmatrix} + t\begin{pmatrix} 10 \\ 0 \\ -6 \end{pmatrix}$$

attempts to find their $D(t)$ ***(M1)***

$D(t) = \sqrt{(10t-18)^2 + 1 + (11-6t)^2}$ ***A1***

OR

attempts to find $\boldsymbol{r}_A - \boldsymbol{r}_B$ ***(M1)***

$$\boldsymbol{r}_A - \boldsymbol{r}_B = \begin{pmatrix} 18 \\ -1 \\ -11 \end{pmatrix} + t\begin{pmatrix} -10 \\ 0 \\ 6 \end{pmatrix}$$

attempts to find their $D(t)$ ***(M1)***

$D(t) = \sqrt{(18-10t)^2 + (-1)^2 + (-11+6t)^2}$ ***A1***

Note: Award ***M0M0A0*** for expressions using two different time parameters.

THEN

either attempts to find the local minimum point of $D(t)$ or attempts to find the value of t such that $D'(t) = 0$ (or equivalent) ***(M1)***

$t = 1.8088... \left(= \frac{123}{68}\right)$

$D(t) = 1.01459...$

minimum value of $D(t)$ is $1.01 \left(= \frac{\sqrt{1190}}{34}\right)$ (km) ***A1***

[5 marks]

Note: Award ***M0*** for attempts at the shortest distance between two lines.

Total [20 marks]

12. (a) rate of growth (change) of the (marsupial) population (with respect to time) ***A1***

[1 mark]

Note: Do not accept growth (change) in the (marsupials) population per year.

(b) **METHOD 1**

attempts implicit differentiation on $\frac{\mathrm{d}P}{\mathrm{d}t} = kP - \frac{kP^2}{N}$ by expanding $kP\left(1 - \frac{P}{N}\right)$ ***(M1)***

$\frac{\mathrm{d}^2P}{\mathrm{d}t^2} = k\frac{\mathrm{d}P}{\mathrm{d}t} - 2\frac{kP}{N}\frac{\mathrm{d}P}{\mathrm{d}t}$ ***A1A1***

$= k\frac{\mathrm{d}P}{\mathrm{d}t}\left(1 - \frac{2P}{N}\right)$ ***A1***

$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{N}\right)$ and so $\frac{\mathrm{d}^2P}{\mathrm{d}t^2} = k^2P\left(1 - \frac{P}{N}\right)\left(1 - \frac{2P}{N}\right)$ ***AG***

METHOD 2

attempts implicit differentiation (product rule) on $\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{N}\right)$ ***M1***

$\frac{\mathrm{d}^2P}{\mathrm{d}t^2} = k\frac{\mathrm{d}P}{\mathrm{d}t}\left(1 - \frac{P}{N}\right) + kP\left(-\left(\frac{1}{N}\right)\frac{\mathrm{d}P}{\mathrm{d}t}\right)$ ***A1***

substitutes $\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{N}\right)$ into their $\frac{\mathrm{d}^2P}{\mathrm{d}t^2}$ ***M1***

$\frac{\mathrm{d}^2P}{\mathrm{d}t^2} = k\left(kP\left(1 - \frac{P}{N}\right)\right)\left(1 - \frac{P}{N}\right) + kP\left(-\left(\frac{1}{N}\right)kP\left(1 - \frac{P}{N}\right)\right)$

$= k^2P\left(1 - \frac{P}{N}\right)^2 - k^2P\left(1 - \frac{P}{N}\right)\left(\frac{P}{N}\right)$

$= k^2P\left(1 - \frac{P}{N}\right)\left(1 - \frac{P}{N} - \frac{P}{N}\right)$ ***A1***

so $\frac{\mathrm{d}^2P}{\mathrm{d}t^2} = k^2P\left(1 - \frac{P}{N}\right)\left(1 - \frac{2P}{N}\right)$ ***AG***

[4 marks]

continued…

Question 12 continued

(c) $\frac{d^2P}{dt^2} = 0 \Rightarrow k^2P\left(1-\frac{P}{N}\right)\left(1-\frac{2P}{N}\right) = 0$ ***(M1)***

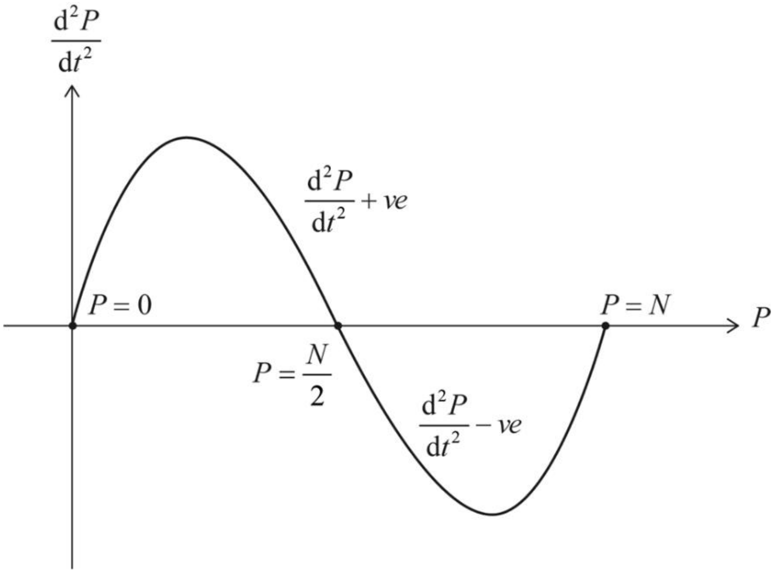
$P = 0, \frac{N}{2}, N$ ***A2***

Note: Award ***A1*** for $P = \frac{N}{2}$ only.

uses the second derivative to show that concavity changes at $P = \frac{N}{2}$ or the first derivative to show a local maximum at $P = \frac{N}{2}$ ***M1***

EITHER

a clearly labelled correct sketch of $\frac{d^2P}{dt^2}$ versus P showing $P = \frac{N}{2}$ corresponding to a local maximum point for $\frac{dP}{dt}$ ***R1***



OR

a correct and clearly labelled sign diagram (table) showing $P = \frac{N}{2}$ corresponding to a local maximum point for $\frac{dP}{dt}$ ***R1***

continued…

Question 12 continued

OR

for example, $\frac{\mathrm{d}^2P}{\mathrm{d}t^2}=\frac{3k^2N}{32}(>0)$ with $P=\frac{N}{4}$ and $\frac{\mathrm{d}^2P}{\mathrm{d}t^2}=-\frac{3k^2N}{32}(<0)$ with $P=\frac{3N}{4}$

showing $P=\frac{N}{2}$ corresponds to a local maximum point for $\frac{\mathrm{d}P}{\mathrm{d}t}$ ***R1***

so the population is increasing at its maximum rate when $P=\frac{N}{2}$ ***AG***

[5 marks]

(d) substitutes $P=\frac{N}{2}$ into $\frac{\mathrm{d}P}{\mathrm{d}t}$ ***(M1)***

$$\frac{\mathrm{d}P}{\mathrm{d}t}=k\left(\frac{N}{2}\right)\left(1-\frac{\frac{N}{2}}{N}\right)$$

the maximum value of $\frac{\mathrm{d}P}{\mathrm{d}t}$ is $\frac{kN}{4}$ ***A1***

[2 marks]

continued…

Question 12 continued

(e) **METHOD 1**

attempts to separate variables **_M1_**

$$\int \frac{N}{P(N-P)}\mathrm{d}P = \int k\ \mathrm{d}t$$

attempts to write $\frac{N}{P(N-P)}$ in partial fractions form **_M1_**

$$\frac{N}{P(N-P)} \equiv \frac{A}{P} + \frac{B}{(N-P)} \Rightarrow N \equiv A(N-P) + BP$$

$A=1,\ B=1$ **_A1_**

$$\frac{N}{P(N-P)} \equiv \frac{1}{P} + \frac{1}{(N-P)}$$

$$\int\left(\frac{1}{P} + \frac{1}{(N-P)}\right)\mathrm{d}P = \int k\ \mathrm{d}t$$

$\Rightarrow \ln P - \ln(N-P) = kt(+C)$ **_A1A1_**

Note: Award ***A1*** for $-\ln(N-P)$ and ***A1*** for $\ln P$ and $kt(+C)$. Absolute value signs are not required.

attempts to find C in terms of N and P_0 **_M1_**

when $t=0$, $P=P_0$ and so $C = \ln P_0 - \ln(N-P_0)$

$$kt = \ln\left(\frac{P}{N-P}\right) - \ln\left(\frac{P_0}{N-P_0}\right)\left(= \ln\left(\frac{\frac{P}{N-P}}{\frac{P_0}{N-P_0}}\right)\right)$$ **_A1_**

so $kt = \ln\frac{P}{P_0}\left(\frac{N-P_0}{N-P}\right)$ **_AG_**

[7 marks]

continued…

Question 12 continued

METHOD 2

attempts to separate variables **_M1_**

$$\int \frac{1}{P\left(1-\frac{P}{N}\right)}\mathrm{d}P=\int k\ \mathrm{d}t$$

attempts to write $\frac{1}{P\left(1-\frac{P}{N}\right)}$ in partial fractions form **_M1_**

$$\frac{1}{P\left(1-\frac{P}{N}\right)}\equiv\frac{A}{P}+\frac{B}{1-\frac{P}{N}}\Rightarrow 1\equiv A\left(1-\frac{P}{N}\right)+BP$$

$A=1,\ B=\frac{1}{N}$ **_A1_**

$$\frac{1}{P\left(1-\frac{P}{N}\right)}\equiv\frac{1}{P}+\frac{1}{N\left(1-\frac{P}{N}\right)}$$

$$\int \frac{1}{P}+\frac{1}{N\left(1-\frac{P}{N}\right)}\mathrm{d}P=\int k\ \mathrm{d}t$$

$\Rightarrow \ln P-\ln\left(1-\frac{P}{N}\right)=kt(+C)$ **_A1A1_**

Note: Award ***A1*** for $-\ln\left(1-\frac{P}{N}\right)$ and ***A1*** for $\ln P$ and $kt(+C)$. Absolute value signs are not required.

continued…

Question 12 continued

$$\ln\left(\frac{P}{1-\frac{P}{N}}\right)=kt+C \Rightarrow \ln\left(\frac{NP}{N-P}\right)=kt+C$$

attempts to find C in terms of N and P_0 ***M1***

when $t=0$, $P=P_0$ and so $C=\ln\left(\frac{NP_0}{N-P_0}\right)$

$$kt=\ln\left(\frac{NP}{N-P}\right)-\ln\left(\frac{NP_0}{N-P_0}\right)\left(=\ln\left(\frac{\frac{P}{N-P}}{\frac{P_0}{N-P_0}}\right)\right)$$ ***A1***

$$kt=\ln\frac{P}{P_0}\left(\frac{N-P_0}{N-P}\right)$$ ***AG***

[7 marks]

continued…

Question 12 continued

METHOD 3

lets $u=\frac{1}{P}$ and forms $\frac{\mathrm{d}u}{\mathrm{d}t}=-\frac{1}{P^2}\frac{\mathrm{d}P}{\mathrm{d}t}$ ***M1***

multiplies both sides of the differential equation by $-\frac{1}{P^2}$ and makes the above substitutions ***M1***

$$-\frac{1}{P^2}\frac{\mathrm{d}P}{\mathrm{d}t}=k\left(\frac{1}{N}-\frac{1}{P}\right)\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}t}=k\left(\frac{1}{N}-u\right)$$

$\frac{\mathrm{d}u}{\mathrm{d}t}+ku=\frac{k}{N}$ (linear first-order DE) ***A1***

$\mathrm{IF}=\mathrm{e}^{\int k\ \mathrm{d}t}=\mathrm{e}^{kt}\Rightarrow\mathrm{e}^{kt}\frac{\mathrm{d}u}{\mathrm{d}t}+k\mathrm{e}^{kt}u=\frac{k}{N}\mathrm{e}^{kt}$ ***(M1)***

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(u\mathrm{e}^{kt}\right)=\frac{k}{N}\mathrm{e}^{kt}$$

$u\mathrm{e}^{kt}=\frac{1}{N}\mathrm{e}^{kt}\left(+C\right)\ \left(\frac{1}{P}\mathrm{e}^{kt}=\frac{1}{N}\mathrm{e}^{kt}\left(+C\right)\right)$ ***A1***

attempts to find C in terms of N and P_0 ***M1***

when $t=0$, $P=P_0$, $u=\frac{1}{P_0}$ and so $C=\frac{1}{P_0}-\frac{1}{N}\left(=\frac{N-P_0}{NP_0}\right)$

$$\mathrm{e}^{kt}\left(\frac{N-P}{NP}\right)=\frac{N-P_0}{NP_0}$$

$\mathrm{e}^{kt}=\left(\frac{P}{N-P}\right)\left(\frac{N-P_0}{P_0}\right)$ ***A1***

$kt=\ln\frac{P}{P_0}\left(\frac{N-P_0}{N-P}\right)$ ***AG***

[7 marks]

continued…

Question 12 continued

(f) substitutes $t=10$, $P=3P_0$ and $N=4P_0$ into $kt=\ln\frac{P}{P_0}\left(\frac{N-P_0}{N-P}\right)$ ***M1***

$10k=\ln 3\left(\frac{4P_0-P_0}{4P_0-3P_0}\right)\ (=\ln 9)$

$k=0.220\ \left(=\frac{1}{10}\ln 9,=\frac{1}{5}\ln 3\right)$ ***A1***

[2 marks]

Total [21 marks]